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LETTER TO THE EDITOR

Infrared Green function for the Coulomb interaction†

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Abstract. Fradkin's representation for the Green function, in an external potential, which has been very useful in field theory, is used to give an intuitive approach in the derivation of the infrared Green function for the Coulomb interaction describing the propagation of the charged particles before and after the collision. The solution is compared with the exact expression for the infrared Green function obtained by Schwinger.

It is well known that long range interactions, such as the Coulomb interaction, lead to so-called modified free plane waves at large distances as a result of the slow decrease of the potentials at such distances (cf Landau and Lifschitz 1965) (for modern treatments, cf Dollard 1964, 1973, Weinberg 1965, Manoukian and Prugovečki 1971, Schweber 1973; see also Kulish and Faddeev 1970, Gervais and Zwanziger 1980, Zwanziger 1975, Korthals Altes and de Rafael 1976, Manoukian 1984). This property is, of course, reflected in the modification of the free Green functions to infrared modified Green functions describing the propagation of the particles *before* and *after* the collision. The latter means that the particles feel the presence of the potential tail even at large distances. The purpose of this letter is to give an extremely simple and an intuitive approach to the derivation of the infrared Green function for the Coulomb interaction by using Fradkin's elegant representation (Fradkin *et al* 1970 and references therein) for a Green function in an external potential. This method has been very useful in field theory (Fradkin *et al* 1970 and references therein). The solution is then compared with the exact expression for the Green function as obtained long ago by Schwinger (1964) based on detailed and complex analysis.

We first consider the free Green function $G_0(x-x')$ which satisfies the differential equation:

$$(-\mathrm{i}\,\partial/\partial t-\nabla^2/2m)G_0(x-x')=\delta^4(x-x'), \tag{1}$$

whose solution is given by

$$G_0(x-x')=\int\frac{(\mathrm{d}k)}{(2\pi)^4}\frac{\exp[\mathrm{i}k(x-x')]}{(-k^0+\boldsymbol{k}^2/2m-\mathrm{i}\varepsilon)},\qquad \varepsilon\to+0 \tag{2}$$

$(\mathrm{d}k)=\mathrm{d}k^0\,\mathrm{d}k^1\,\mathrm{d}k^2\,\mathrm{d}k^3$, $kx=\boldsymbol{k}\cdot\boldsymbol{x}-k^0x^0$, $x^0=t$, or

$$G_0(x-x')=\mathrm{i}\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\exp[\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')]\exp[-\mathrm{i}\boldsymbol{k}^2(x^0-x^{0\prime})/2m],\qquad \text{for } x^0>x^{0\prime}, \tag{3}$$

$$G_0(x-x')=0,\qquad \text{for } x^0<x^{0\prime}. \tag{4}$$

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In the presence of a potential $V(\boldsymbol{x})$, the exact Green function $G(x, x')$ satisfies the equation

$$(-\mathrm{i}\,\partial/\partial t - \nabla^2/2m + V(\boldsymbol{x}))G(x, x') = \delta^4(x - x'), \tag{5}$$

where we suppose that

$$V(\boldsymbol{x}) = \mathrm{O}(|\boldsymbol{x}|^{-\alpha}), \qquad |\boldsymbol{x}| \to \infty. \tag{6}$$

For $\alpha > 1$, the potential is termed as short range (see equation (14)). The interesting case is when $\alpha = 1$, corresponding to the Coulomb interaction.

It is convenient to introduce the transform

$$G(x, x') = \int \frac{(\mathrm{d}p)}{(2\pi)^4} G(\boldsymbol{x}, p) \exp[\mathrm{i}p(x - x')], \tag{7}$$

where $G(\boldsymbol{x}, p)$ satisfies the differential equation

$$(-p^0 + \boldsymbol{p}^2/2m - \mathrm{i}\boldsymbol{p}\cdot\nabla/m - \nabla^2/2m + V(\boldsymbol{x}))G(\boldsymbol{x}, p) = 1. \tag{8}$$

For a free particle, $G(\boldsymbol{x}, p)$ is independent of $\boldsymbol{x}$ (no scattering). Accordingly, we will seek an expression for the infrared Green function $G^0(p)$ which is independent of $\boldsymbol{x}$, and at the same time, correctly takes into account the slow decrease of the Coulomb potential at large distances. To this end, we write (Fradkin *et al* 1970)

$$G(\boldsymbol{x}, p) = \mathrm{i}\int_0^\infty \mathrm{d}\nu \exp[-\mathrm{i}\nu(-p^0 + \boldsymbol{p}^2/2m - \mathrm{i}\varepsilon)]F(\nu). \tag{9}$$

The parameter ν has the dimension of time. Accordingly, we expect that if $\boldsymbol{p}$ is the momentum of a particle emerging from the scattering centre, then the particle is essentially free, and we have the following approximate relation holding after (and before) the collision (e.g. Dollard and Velo 1966) based on a dimensional argument:

$$|\boldsymbol{x}|/\nu \sim |\boldsymbol{p}|/m, \qquad \nu \to \infty. \tag{10}$$

Therefore, to obtain an $\boldsymbol{x}$-independent (no scattering) solution of (8), we replace $\boldsymbol{x}$ in $V(\boldsymbol{x}) \sim V_{\text{asym.}}(|\boldsymbol{x}|)$, for $|\boldsymbol{x}| \to \infty$, by $\nu|\boldsymbol{p}|/m$. Equation (8) then leads to the elementary differential equation ($F(0) = 1$)

$$\mathrm{i}\,\partial F(\nu)/\partial\nu = V_{\text{asym.}}(\nu|\boldsymbol{p}|/m)F(\nu), \qquad \nu \to \infty, \tag{11}$$

whose solution is

$$F(\nu) \sim \exp\left(-\mathrm{i}\int \mathrm{d}\nu\, V_{\text{asym.}}(\nu|\boldsymbol{p}|/m)\right) \qquad \nu \to \infty. \tag{12}$$

Hence from equation (6), with $V_{\text{asym.}}(|\boldsymbol{x}|) = C|\boldsymbol{x}|^{-\alpha}$, we have

$$F(\nu) \sim \begin{cases} \exp\left(\dfrac{-\mathrm{i}Cm^\alpha}{|\boldsymbol{p}|^\alpha}\dfrac{\nu^{1-\alpha}}{(1-\alpha)}\right) \sim 1, & \nu \to \infty, \quad \alpha > 1 \\ \exp\left(\dfrac{-\mathrm{i}Cm}{|\boldsymbol{p}|}\ln(\nu)\right), & \nu \to \infty, \quad \alpha = 1. \end{cases} \tag{13}$$

From (13) and (9) we obtain the modified free (infrared) Green function (in p-space)

$$G^0(p) \sim \begin{cases} (-p^0 + \boldsymbol{p}^2/2m)^{-1}, & \alpha > 1 \\ \left(-p^0 + \dfrac{\boldsymbol{p}^2}{2m}\right)^{-1} \exp\left(\dfrac{\mathrm{i}Cm}{|\boldsymbol{p}|}\ln\left(p^0 - \dfrac{\boldsymbol{p}^2}{2m}\right)\right), & \alpha = 1. \end{cases} \tag{14}$$

Hence, for short range interactions, the free Green function is not modified, but a similar logarithmic factor arises for the Coulomb interaction. For the latter $V(\boldsymbol{x}) = Ze^2/|\boldsymbol{x}|$, and from (14)

$$G^0(p) \sim \left(-p^0 + \frac{\boldsymbol{p}^2}{2m}\right)^{-1} \exp\left(\mathrm{i}\frac{Ze^2 m}{|\boldsymbol{p}|} \ln\left(p^0 - \frac{\boldsymbol{p}^2}{2m}\right)\right). \tag{15}$$

This should be compared with the exact infrared Green function (near the energy shell) obtained for example in Schwinger (1964):

$$G^0(p) = \frac{H(p)^{1/2}}{(-p^0 + \boldsymbol{p}^2/2m)} \exp\left[\mathrm{i}\frac{Ze^2 m}{|\boldsymbol{p}|} \ln\left(\left(p^0 - \frac{\boldsymbol{p}^2}{2m}\right) \Big/ 4p^0\right)\right],$$
$$H(p) = (2\pi Ze^2 m/|\boldsymbol{p}|)/(\exp(2\pi Ze^2 m/|\boldsymbol{p}|) - 1) \tag{16}$$

and the latter is generally of order 1. The propagator $G^0(p)$ describes the propagation of a particle before and after the collision. It is a function of the coupling Ze^2. That is, the particles feel the presence of the interaction even at large distances. On the other hand, for short range potentials (including the Yukawa potential) the free propagators are not modified (see equation (14)).

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